

\mathcal{L} -invariant for Siegel-Hilbert forms

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We prove in some cases a formula for the Greenberg-Benois \mathcal{L} -invariant of the spin, standard and adjoint Galois representations associated with Siegel-Hilbert modular forms. In order to simplify the calculation, we give a new definition of the \mathcal{L} -invariant for a Galois representation V of a number field $F \neq \mathbb{Q}$; we also check that it is compatible with Benois' definition for $\text{Ind}_F^{\mathbb{Q}}(V)$.

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1 Introduction

Since the historical results of Kummer and Kubota-Leopold on congruences for Bernoulli numbers, people have been interested in studying the p -adic variation of special values of L -functions.

More precisely, fix a motive M over \mathbb{Q} . We suppose that M is Deligne critical at $s = 0$ and that there exists a Deligne's period $\Omega(M)$ such that $\frac{L(M, 0)}{\Omega(M)}$ is algebraic. Fix a prime p and two embeddings

$$\mathbb{C}_p \hookleftarrow \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}.$$

Let V be the p -adic realization of M and suppose that V is semistable (à la Fontaine). Thanks to work of Coates and Perrin-Riou, we have now days precise conjectures on how the special values should behave p -adically; we fix a regular sub-module of V . This corresponds to the choice of a sub- (φ, N) -module of $\mathcal{D}_{\text{st}}(V)$ which is a section of the exponential map

$$\mathcal{D}_{\text{st}}(V) \rightarrow t(V) \cong \frac{\mathcal{D}_{\text{st}}(V)}{\text{Fil}^0 \mathcal{D}_{\text{st}}(V)}.$$

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Let h be the valuation of the determinant of φ on D . We can state the following conjecture

Conjecture 1.1. *There exists a formal series $L_p^D(V, T) \in \mathbb{C}_p[[T]]$ who grows as \log_p^h such that for all non-trivial, finite-order characters $\varepsilon : 1 + p\mathbb{Z}_p \rightarrow \mu_{p^\infty}$ we have*

$$L_p^D(V, \varepsilon(1+p) - 1) = C_\varepsilon(D) \frac{L(M, 0)}{\Omega(M)}.$$

Moreover, for $\varepsilon = \mathbf{1}$ we have

$$L_p(V, 0) = E(D) \frac{L(M, 0)}{\Omega(M)},$$

where $E(D)$ is an explicit product of Euler-type factors depending on D and $(\mathcal{D}_{\text{st}}(V)/D)^{N=0}$.

It may happen that one of the factor of $E(D)$ vanishes and then we say that trivial zeros appear. Since the seminal work of [MTT86], people have been interested in describing the p -adic derivative of $L_p^D(V, (1+p)^s - 1)$ when trivial zeros appear.

We suppose for simplicity that $L(M, 0)$ is not vanishing. We have the following conjecture by Greenberg and Benois;

Conjecture 1.2. *Let t the number of vanishing factors of $E(D)$. Then*

- $\text{ord}_{s=0} L_p^D(V, (1+p)^s - 1) = t,$
- $L_p(V, 0)^* = \mathcal{L}(V, D) E^*(D) \frac{L(M, 0)}{\Omega(M)}.$

Here $E^*(D)$ is the product of non-vanishing factors of $E(D)$ and $\mathcal{L}(V, D)$ is a number defined in purely Galois theoretical terms (see Section 3.1).

The error factor $\mathcal{L}(V, D)$ is quite mysterious. It has been calculated in only few cases for the symmetric square of a (Hilbert) modular form by Hida, Mok and Benois and for symmetric power of Hilbert modular forms by Hida and Jorza-Harron. Unless V is an elliptic curve over \mathbb{Q} with multiplicative reduction at p we can not prove the non-vanishing of $\mathcal{L}(V, D)$.

The aim of this paper is to calculate it in some new cases; let F be a totally real field where p is unramified and π be an automorphic representation of $\text{GSp}_{2g/F}$. We suppose that it has Iwahoric level at all $\mathfrak{p} \mid p$. We suppose moreover that $\pi_{\mathfrak{p}}$ is either Steinberg (see Definition 4.9) or spherical. We partition consequently the prime ideals of F above p in $S^{\text{Stb}} \cup S^{\text{Sph}}$.

We have conjecturally two Galois representations associated to π , namely the spinorial one V_{spin} and the standard one V_{sta} . Let V be one of these two representations. We choose for each prime \mathfrak{p} of F dividing p a regular sub module $D_{\mathfrak{p}}$ of $\mathcal{D}_{\text{st}}(V|_{G_{F_{\mathfrak{p}}}})$.

Consider a family of Siegel-Hilbert modular forms as in [Urb11] passing through π . Let us denote by $\beta_{\mathfrak{p}}(\kappa)$ the eigenvalue of the normalized Hecke operators $U_{1, \mathfrak{p}}$ (see Definition 4.10) on this family. Let $S^{\text{Sph}, 1} = S^{\text{Sph}, 1}(V, D)$ be the subset of S^{Sph} for which $(\mathcal{D}_{\text{st}}(V_{\mathfrak{p}})/D_{\mathfrak{p}})^{N=0}$ does not contain the eigenvalue 1. Conjecturally, it is empty for the spin representation. The eigenvalues 1 always appears in $\mathcal{D}_{\text{st}}(V_{\mathfrak{p}})$ for V the standard representation but it may appear in $D_{\mathfrak{p}}$ (this is already the case for the symmetric square of a modular form).

Let t_{Stb} be the cardinality of S^{Stb} and t_{Sph} be the cardinality of $S^{\text{Sph}, 1}$. We define $f_{\mathfrak{p}} = [F_{\mathfrak{p}}^{\text{ur}} : \mathbb{Q}_p]$.

Theorem 1.3. *Let π be as above, of parallel weight \underline{k} . Let $V = V_{\text{spin}}$ and suppose hypothesis **LGp** of Section 4.2, then the expected number of trivial zero for $L_p^D(V(k-1), T)$ is t_{Stb} and*

$$\mathcal{L}(V(k-1), D) = \prod_{\mathfrak{p} \in S^{\text{Stb}}} -\frac{1}{f_{\mathfrak{p}}} \frac{d \log_p \beta_{\mathfrak{p}}(k)}{dk} \Big|_{k=\underline{k}}.$$

Let $V = V_{\text{std}}$, then the conjectural number of trivial zero for $L_p^D(V, T)$ is $t_{\text{Stb}} + t_{\text{Sph}}$ and

$$\mathcal{L}(V, D) = \mathcal{L}(V, D)^{\text{Sph}} \prod_{\mathfrak{p} \in S^{\text{Stb}}} -\frac{1}{f_{\mathfrak{p}}} \frac{d \log_p \beta_{\mathfrak{p}}(k)}{dk} \Big|_{k=\underline{k}},$$

where $\mathcal{L}(V, D)^{\text{Sph}}$ is a priori global factor. It is 1 if $t_{\text{Sph}} = 0$.

In Section 4.2 we shall provide also a formula for the \mathcal{L} -invariant of $V_{\text{std}}(s)$ ($\min(k - g - 1, g - 1) \geq s \geq 1$).

The proof of the theorem is not different from the one of [Ben10, Theorem 2] which in turn is similar to the original one of [GS93].

Let now $g = 2$. Let t be the number of primes above p in F . We consider the $2t$ -dimensional eigenvariety for $\text{GSp}_{4/F}$ with variables $k = \{k_{\mathfrak{p},1}, k_{\mathfrak{p},2}\}_{\mathfrak{p}}$ (see Section 5) and let us denote by $F_{\mathfrak{p},i}(k)$ ($i = 1, 2$) the first two graded pieces of $\mathbf{D}_{\text{rig}}^{\dagger}(V_{\text{spin}})$. The 10-dimensional Galois representation $\text{Ad}(V_{\text{spin}})$ has a natural regular sub- (φ, N) -module induced by the p -refinement of $\mathbf{D}_{\text{rig}}^{\dagger}(V_{\text{spin}})$ and which we shall denote by D_{Ad} . With this choice of regular sub module, $\text{Ad}(V_{\text{spin}})$ presents $2t$ trivial zeros. In Section 5 we prove the following theorem;

Theorem 1.4. *Let π be an automorphic form of weight \underline{k} and suppose hypothesis **LGp** of Section 4.2 is verified for V_{spin} , we have then*

$$\mathcal{L}(\text{Ad}(V_{\text{spin}}(\pi)), D_{\text{Ad}}) = \prod_{\mathfrak{p}} \frac{2}{f_{\mathfrak{p}}^2} \det \begin{pmatrix} \frac{\partial \log_p F_{\mathfrak{p},1}(k)}{\partial k_{\mathfrak{p},1}} & \frac{\partial \log_p F_{\mathfrak{p},2}(k)}{\partial k_{\mathfrak{p},1}} \\ \frac{\partial \log_p F_{\mathfrak{p},1}(k)}{\partial k_{\mathfrak{p},2}} & \frac{\partial \log_p F_{\mathfrak{p},2}(k)}{\partial k_{\mathfrak{p},2}} \end{pmatrix} \Big|_{1 \leq i,j \leq t |_{k=\underline{k}}}.$$

We remark that this theorem is the first to really go beyond GL_2 and its representations Sym^n .

The motivation for Theorem 1.3 lies in a generalization of [Ros13b] to Siegel forms. In *loc. cit.* we use Greenberg-Stevens method to prove a formula for the derivative of the symmetric square p -adic L -function and calculate the analytic \mathcal{L} -invariant and the same method of proof can be generalized to finite slope Siegel forms thanks to the overconvergent Maß-Shimura operators and overconvergent projectors of Z. Liu's thesis. With some work, it could also be generalized to totally real field where p where is inert, as already done for the symmetric square [Ros13a].

We hope to calculate the \mathcal{L} -invariant for V_{std} and $\text{Ad}(V_{\text{spin}})$ for more general forms in a future work.

In Section 2 we recall the theory of (φ, Γ) -module over a finite extension of \mathbb{Q}_p . It will be used in Section 3 to generalize the definition of the \mathcal{L} -invariant à la Greenberg-Benois to Galois representations V over general number field F (note that we do not suppose p split or unramified). This definition does not require one to pass through $\text{Ind}_F^{\mathbb{Q}}(V)$ to calculate the \mathcal{L} -invariant which in turn simplifies explicit calculation. We shall check that this definition coincides with Benois' definition for $\text{Ind}_F^{\mathbb{Q}}(V)$.

We prove the above-mentioned theorems in Section 4 and 5, inspired mainly by the methods of [Hid07].

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2 Some results on rank one (φ, Γ) -module

Let L be a finite extension of \mathbb{Q}_p . The aim of this section is to recall certain results concerning (φ, Γ) -modules over the Robba ring \mathcal{R}_L . Let L_0 be the maximal unramified extension contained in L . Let L'_0 be the maximal unramified extension contained in $L_\infty := L(\mu_{p^\infty})$ and $L' = L \cdot L'_0$. Let $e_L := [L(\mu_{p^\infty}) : L_0(\mu_{p^\infty})] = [\Gamma_{\mathbb{Q}_p} : \Gamma_L]$, where $\Gamma_L := \text{Gal}(L_\infty/L)$. We define

$$\mathbf{B}_{L, \text{rig}}^{\dagger, r} = \left\{ f = \sum_{n \in \mathbb{Z}} a_n \pi_L^n \mid a_n \in L'_0, \text{ such that } f(X) = \sum_{n \in \mathbb{Z}} a_n X^n \right. \\ \left. \text{is holomorphic on } p^{-\frac{1}{e_L r}} \leq |X|_p < 1 \right\}, \\ \mathcal{R}_L = \bigcup_r \mathbf{B}_{K, \text{rig}}^{\dagger, r},$$

where π_L is a certain uniformizer coming from the theory of field of norm. We have an action of φ on \mathcal{R}_L . If $L = L_0$, there is no ambiguity and we have:

$$\varphi(\pi_L) = (1 + \pi_L)^p - 1, \quad \varphi(a_n) = \varphi_{L'_0}(a_n).$$

Otherwise the action on π_L is more complicated. Similarly, we have a Γ_L -action. If $L = L_0$ we have

$$\gamma(\pi_L) = (1 + \pi_L)^{\chi_{\text{cycl}}(\gamma)} - 1,$$

where χ_{cycl} is the cyclotomic character. If L is ramified we also have an action of Γ_L on the coefficients given by

$$\gamma(a_n) = \sigma_\gamma(a_n)$$

where σ_γ is the image of γ via

$$\Gamma_L \rightarrow \Gamma_L / \Gamma_{L'} \xrightarrow{\cong} \text{Gal}(L'_0/L_0).$$

If a_n is fixed by φ and Γ_L , then is it in \mathbb{Q}_p . We have $\text{rk}_{\mathcal{R}_{\mathbb{Q}_p}} \mathcal{R}_L = [L_\infty : \mathbb{Q}_{p, \infty}]$.

Let $\delta : L^\times \rightarrow E^\times$ be a continuous character. We define $\mathcal{R}_L(\delta)$ to be the rank one (φ, Γ_L) -module with basis e_δ for which $\varphi(e_\delta) = \delta(\pi_L)e_\delta$ and $\gamma(e_\delta) = \delta(\chi_{\text{cycl}}(\gamma))e_\delta$.

We classify now the cohomology of such a (φ, Γ_L) -modules. It will be useful to calculate it explicitly in terms of $C_{\varphi, \gamma}$ -complexes [Ben11, §1.1.5]. We fix then a generator γ_L of Γ_L ; if clear from the context, we shall drop the subscript $_L$ and write simply γ .

Proposition 2.1. *We have $H^0(\mathcal{R}_L(\delta)) = 0$ unless $\delta(z) = \prod_\tau \tau(z)^{m_\tau}$ with $m_\tau \leq 0$ for all τ ; in this case we have $H^0(\mathcal{R}_L \delta) \cong E$. We shall denote its basis by $t^m \otimes e_\delta$, where*

$$t^m = \oplus t^{m_\tau} \in \oplus_\tau B_{\text{dR}}^+ \otimes_{L, \sigma} E.$$

If $\delta(z) = \prod_\tau \tau(z)^{m_\tau}$ with $m_\tau \leq 0$, then

$$\dim_E H^1(\mathcal{R}_L(\delta)) = [L : \mathbb{Q}_p] + 1.$$

If $\delta(z) = |N_{L/\mathbb{Q}_p}(z)|_p \prod_\tau \tau(z)^{k_\tau}$ with $k_\tau \geq 1$, then

$$\dim_E H^1(\mathcal{R}_L(\delta)) = [L : \mathbb{Q}_p] + 1.$$

Otherwise

$$\dim_E H^1(\mathcal{R}_L(\delta)) = [L : \mathbb{Q}_p].$$

We have $H^2(\mathcal{R}_L(\delta)) = 0$ unless $\delta(z) = |\mathbb{N}_{L/\mathbb{Q}_p}(z)|_p \prod_{\tau} \tau(z)^{k_{\tau}}$ with $k_{\tau} \geq 1$; in this case we have $H^2(\mathcal{R}_L(\delta)) \cong E$.

Note that when we choose t^m as a basis we are implicitly using the fact that we can embed certain sub-rings of \mathcal{R}_L into B_{dR}^+ (see [Ben11, §1.2.1]).

Proof. The same results is stated in [Nak09, Proposition 2.14, 2.15] for $E - B$ -pairs, but the proof for (φ, Γ) -modules is the same.

Recall that have a canonical duality [Liu08] given by cup product

$$H^i(D) \times H^{2-i}(D^*(\chi_{\text{cycl}})) \rightarrow H^2(\chi_{\text{cycl}}).$$

The last fact is then a direct consequence. □

This allows us to define a canonical basis of $H^2(\mathcal{R}_L(|\mathbb{N}_{L/\mathbb{Q}_p}(z)|_p \prod_{\tau} \tau(z)^{k_{\tau}}))$. We define $H_{\text{f}}^1(D)$ has the H^1 of the complex

$$\mathcal{D}_{\text{cris}}(D) \rightarrow t_D \oplus \mathcal{D}_{\text{cris}}(D)$$

and we have immediately [Nak09, Proposition 2.7]

$$\dim_E H_{\text{f}}^1(D) = \dim_E(H^0(D)) + \dim_E t_D. \quad (2.2)$$

Hence

Lemma 2.3. *If $\delta(z) = \prod_{\tau} \tau(z)^{m_{\tau}}$ with $m_{\tau} \leq 0$, then*

$$\dim_E H_{\text{f}}^1(\mathcal{R}_L(\delta)) = 1.$$

If $\delta(z) = |\mathbb{N}_{L/\mathbb{Q}_p}(z)|_p \prod_{\tau} \tau(z)^{k_{\tau}}$ with $k_{\tau} \geq 1$, then

$$\dim_E H_{\text{f}}^1(\mathcal{R}_L(\delta)) = d.$$

We now want to calculate $H_{\text{f}}^1(\mathcal{R}_L(\delta))$ for $\delta(z) = \prod_{\tau} \tau(z)^{m_{\tau}}$ with $m_{\tau} \leq 0$. We recall the following lemma [Ben11, Lemma 1.4.3]

Lemma 2.4. *The extension in $H^1(\mathcal{R}_L(\delta))$ corresponding to the couple (a, b) is crystalline if and only if the equation $(1 - \gamma)x = b$ has a solution in $D[\frac{1}{t}]$*

The following proposition is an immediate consequence of the above lemma [Ben11, Theorem 1.5.7 (i)] (see also the construction of [Nak09] at page 900)

Proposition 2.5. *Let e_{δ} be a basis for $\mathcal{R}_L(\delta)$. Then $x_m = \text{cl}(t^m, 0)e_{\delta}$ is a basis of $H_{\text{f}}^1(\mathcal{R}_L(\delta))$.*

Remark 2.6. *If δ is the trivial character then x_0 corresponds (via class field theory) to the unramified \mathbb{Z}_p -extension of $\text{Hom}(G_L, E^{\times}) \cong H^1(G_L, E)$.*

We have now to cut out a “canonical” one-dimensional subspace in $H^1(\mathcal{R}_L(\delta))$ which trivially intersects $H_{\text{f}}^1(\mathcal{R}_L(\delta))$ (and reduces to the cyclotomic \mathbb{Z}_p -extension in the sense of the previous remark).

We introduce another extension. We define $y_k = \frac{1}{e_L} \log(\chi_{\text{cycl}}(\gamma_L)) \text{cl}(0, t^m)e_{\delta}$.

We explain why this cocycle is of interest for us. We can calculate cohomology of induced $(\varphi, \Gamma_{\mathbb{Q}_p})$ -module. Indeed, we consider now two p -adic fields K and L , L a finite extension of K . The main reference for this part is [Liu08]. Let D be a (φ, Γ_L) -module, we define

$$\mathrm{Ind}_{\Gamma_L}^{\Gamma_K}(D) = \{f : \Gamma_K \rightarrow D \mid f(hg) = hf(g) \ \forall h \in \Gamma_L\}.$$

It has rank $[L : K] \mathrm{rk}_{\mathcal{R}_L}(D)$ over \mathcal{R}_K ; indeed \mathcal{R}_L is a \mathcal{R}_K -module of rank $[L : K]/|\Gamma_K/\Gamma_L|$. (The unramified part of L/K plus the ramified part which is disjoint by K_∞ . See after [Liu08, Theorem 2.2].) If D comes from a G_L -representation V we have

$$\mathbf{D}_{\mathrm{rig}}^\dagger(\mathrm{Ind}_{G_L}^{G_K}(V)) = \mathrm{Ind}_{\Gamma_L}^{\Gamma_K}(\mathbf{D}_{\mathrm{rig}}^\dagger(V)).$$

We have then the equivalent of Shapiro's lemma

$$H^i(D) \cong H^i(\mathrm{Ind}_{\Gamma_L}^{\Gamma_K}(D)).$$

Moreover, the aforementioned duality for (φ, Γ) -modules is compatible with induction [Liu08, Theorem 2.2]. If $D \cong \mathcal{R}_L(\delta)$ is free of rank one, then we have an explicit description of $\mathrm{Ind}_{\Gamma_L}^{\Gamma_K}(D)$. Let $e_\infty = |\Gamma_K/\Gamma_L|$, we write $\{\omega^i\}_{i=0}^{e_\infty-1}$ for $(\Gamma_K/\Gamma_L)^\wedge$. The $\mathrm{Ind}_{\Gamma_L}^{\Gamma_K}(D)$ is the \mathcal{R}_L -span of f_i , where $f_i(g) = \omega^i(g)\delta(\chi_{\mathrm{cycl}}(g))e_\delta$.

We go back to the previous setting, where $K = \mathbb{Q}_p$ (hence $e_\infty = e_L$). We have the following exact sequence

$$0 \rightarrow \mathcal{R}_{\mathbb{Q}_p}(z^{\sum m_\sigma}) \rightarrow \mathrm{Ind}_L^{\mathbb{Q}_p}(\mathcal{R}_L(\delta)) \rightarrow D' \rightarrow 0, \quad (2.7)$$

where we send $e_{z^{\sum m_\tau}}$ to e_δ (the image is $\mathcal{R}_{\mathbb{Q}_p}e_\delta$). Note that $z^{\sum m_\tau}$ is the restriction of δ to \mathbb{Q}_p^* . Note that $H^0(D') = 0$ by calculation similar to [Col08, Proposition 2.1] or [Nak09, Proposition 2.14]. The long exact sequence in cohomology induces

$$\begin{aligned} H^0(\mathcal{R}_{\mathbb{Q}_p}(z^{\sum m_\tau})) &\xrightarrow{\cong} H^0(\mathrm{Ind}_L^{\mathbb{Q}_p}(\mathcal{R}_L(\delta))), \\ H^1(\mathcal{R}_{\mathbb{Q}_p}(z^{\sum m_\tau})) &\hookrightarrow H^1(\mathrm{Ind}_L^{\mathbb{Q}_p}(\mathcal{R}_L(\delta))). \end{aligned}$$

The first morphism is the identity: the fixed basis $t^{\sum m_\tau}e_{z^{\sum m_\tau}}$ is sent to the basis $t^{k_\delta}e_\delta$ (because $E \otimes_{\mathbb{Q}_p} L \cong \oplus_\tau E$).

The cocycle x_m is the image of $x_{\sum m_\tau}$. The image of $y_{\sum m_\tau} \in H_c^1(\mathcal{R}_{\mathbb{Q}_p}(z^{\sum k_\sigma}))$ is instead y_m (if $\gamma_L = \gamma_{\mathbb{Q}_p}^{e_L}$).

Proposition 2.8. *Let D a (φ, Γ) -module over \mathcal{R}_L with non-negative Hodge-Tate weight. Suppose that $\mathcal{D}_{\mathrm{st}}(D) = \mathcal{D}_{\mathrm{st}}(D)^{\varphi=1}$. Then D is crystalline and*

$$D \cong \oplus \mathcal{R}_L(\delta_i)$$

with $\delta_i(z) = \prod_\tau \tau(z)^{k_{i,\tau}}$.

Proof. We follow closely the proof [Ben11, Proposition 1.5.8]. As $N\varphi = p\varphi N$ we obtain immediately that $N = 0$, hence D is crystalline.

Let r be the rank of D over \mathcal{R}_L . We write the Hodge-Tate weight as $(k_i)_{i=1}^r$ where $k_i = (k_{i,\tau})_\tau$ and $k_i \leq k_{i+1}$. We prove it by induction; the case $r = 1$ is clear.

For $r = 2$ we can suppose $k_1 = 0$ by twisting. Let δ be defined by $\prod_\tau \tau(z)^{k_\tau}$. So we have an extension of $\mathcal{R}_L(\delta)$ by \mathcal{R}_L . Let m_2 be a lift to D of a basis of \mathcal{R}_1 . As $\varphi = 1$ we have $\varphi m_2 = m_2$. As the extension is crystalline we know that γ acts trivially too, hence the extension splits.

Suppose now $r > 2$. Take v in $\mathrm{Fil}^{k_d} \mathcal{D}_{\mathrm{st}}$. Define δ_d by $\delta_d(z) = \prod_\tau \tau(z)^{k_{d,\tau}}$, we have

$$0 \rightarrow \mathcal{R}_L(\delta_d) \rightarrow D \rightarrow D' \rightarrow 0.$$

By inductive hypothesis $D' \cong \oplus_{i=1}^{d-1} \mathcal{R}_L(\delta_i)$. We can write

$$\mathrm{Ext}(D', \mathcal{R}_L(\delta_d)) = \oplus_{i=1}^{d-1} \mathrm{Ext}(\mathcal{R}_L(\delta_i), \mathcal{R}_L(\delta_d))$$

and we are reduced to the case $r = 2$ which has already been dealt. \square

We consider a (φ, Γ) -module M which sits in the non-split exact sequence

$$0 \rightarrow M_0 := \bigoplus_{i=1}^r \mathcal{R}_L(\delta_i) \rightarrow M \rightarrow M_1 := \bigoplus_{i=1}^r \mathcal{R}_L(\delta'_i) \rightarrow 0, \quad (2.9)$$

where $\delta_i(z) = |N_{L/\mathbb{Q}_p}(z)|_p \prod_{\tau} \tau(z)^{m_{i,\tau}}$ with $m_{i,\tau} \geq 1$ for all τ and $\delta'_i(z) = \prod_{\tau} \tau(z)^{k_{i,\tau}}$ with $k_{i,\tau} \leq 0$ for all τ . We say that M is of type $U_{m,k}$ if the image of M in $H^1(M_1)$ is crystalline.

Proposition 2.10. *Suppose that M is not of type $U_{m,k}$. Then we have $\dim_E(H^1(M)) = 2[L : \mathbb{Q}_p]r$ and $H^2(M) = H^0(M) = 0$. Moreover, if we write*

$$0 \rightarrow H^0(M_1) \xrightarrow{\Delta_0} H^1(M_0) \xrightarrow{f_1} H^1(M) \xrightarrow{g_1} H^1(M_1) \xrightarrow{\Delta_1} H^2(M_0) \rightarrow 0$$

we have $H^1(M_0) = \text{Im}(\Delta_1) \oplus H_f^1(M_0)$, $\text{Im}(f_1) = H_f^1(M)$ and $H^1(M_1) = \text{Im}(g_1) \oplus H_f^1(M_1)$.

Proof. We have $H^0(M) = 0$. Note that $M^*(\chi_{\text{cycl}})$ is a module of the same type, hence $H^2(M) = H^0(M^*(\chi_{\text{cycl}})) = 0$. We can write

$$0 \rightarrow H^0(M_1) \rightarrow H^1(M_0) \xrightarrow{f_1} H^1(M) \xrightarrow{g_1} H^1(M_1) \rightarrow H^2(M_0) \rightarrow 0$$

and conclude by Proposition 2.1.

Note that $\dim_E H_f^1(M) = rd$ by (2.2).

By hypothesis, we have that $\text{Im}(\Delta_1) \cap H_f^1(M_0) = 0$ and the first statement follows from dimension counting. The third statement follows from duality.

For the second statement $H_f^1(M_0)$ injects into $H_f^1(M)$. As both have the same dimension, we conclude. \square

Suppose now that $M_0 = \mathcal{R}_L(|N_{L/\mathbb{Q}_p}(z)|_p \prod_{\tau} z^{k_{\tau}})$ and $M_1 = \mathcal{R}_L(\prod_{\tau} z^{m_{\tau}})$, we give the following key proposition for the definition of the \mathcal{L} -invariant

Lemma 2.11. *The intersection of $T := \text{Im}(H^1(M))$ and $\text{Im}(H^1(\mathcal{R}_{\mathbb{Q}_p}(z^{\sum_{\tau} m_{\tau}})))$ in $\text{Im}(H^1(M_1))$ is one dimensional.*

Proof. The intersection is non-empty as the sum of their dimension is $d+2$ and $\text{Im}(H^1(M_1))$ has dimension $d+1$. We have that $H_f^1(M_1)$ is contained in $\text{Im}(H^1(\mathcal{R}_{\mathbb{Q}_p}(z^{\sum_{\tau} m_{\tau}})))$ and by the previous proposition the former is not in the image of g_1 and we are done. \square

In particular, we deduce that T surjects into $\text{Im}(H_c^1(\mathcal{R}_{\mathbb{Q}_p}(z^{\sum_{\tau} m_{\tau}})))$.

3 \mathcal{L} -invariant over number fields

Let F be a number field. We consider a global Galois representation

$$V : G_F \rightarrow \text{GL}_n(E)$$

where E is p -adic field. We suppose that it is unramified outside a finite number of places S containing all the p -adic places. We suppose moreover that it is semistable at all places above p (i.e. $\mathcal{D}_{\text{st}}(V|_{F_p})$ is of rank n over $F_p^{\text{ur}} \otimes_{\mathbb{Q}_p} E$, being F_p^{ur} the maximal unramified extension of \mathbb{Q}_p contained in F_p^{ur}).

In this section we generalize Greenberg-Benois definition of the \mathcal{L} -invariant to such a V when it presents trivial zeros. Note that we do not require p split or unramified in F .

Let t be the number of trivial zeros. The classical definition by Greenberg [Gre94] defines the \mathcal{L} -invariant as the “slope” of a certain t -dimension subspace of $H^1(G_{\mathbb{Q}_p}, \mathbb{Q}_p^t)$ which is a $2t$ -dimensional space with a canonical basis given by ord_p and \log_p .

In our setting, the main obstacle is that the cohomology of the trivial (φ, Γ) -module \mathcal{R}_{F_p} is no longer two-dimensional and it is not immediate to find a suitable subspace. Inspired by Hida’s work for symmetric

powers of Hilbert forms [Hid07], we consider the image of $H^1(\mathcal{R}_{\mathbb{Q}_p})$ inside $H^1(\mathcal{R}_{F_p})$.

If t denotes the number of expected trivial zeros, we show that we can define, similarly to [Ben11], a t -dimensional subspace of $H^1(G_{\mathbb{Q},S}, V)$ whose image in $H^1(\mathcal{R}_{\mathbb{Q}_p})$ has trivial intersection with the crystalline cocycle. This is enough to define the \mathcal{L} -invariant; we further check that our definition is compatible with Benois'.

3.1 Definition of the \mathcal{L} -invariant

We define local cohomological conditions L_v in order to define a Selmer group; we denote by G_v a fixed decomposition group at v in $G_{F,S}$ and by I_v the inertia. For $v \nmid p$ we define

$$L_v := \text{Ker} \left(H^1(G_v, V) \rightarrow H^1(I_v, V) \right).$$

If $v \mid p$ we define

$$L_v := H_f^1(F_v, V) = \text{Ker} \left(H^1(G_v, V) \rightarrow H^1(G_v, V \otimes_E \mathbf{B}_{\text{cris}}) \right).$$

If $\mathbf{D}_{\text{rig}}^\dagger(V)$ denotes the (φ, Γ) -module associated with V we also have $L_p = H_f^1(\mathbf{D}_{\text{rig}}^\dagger(V))$. We define then the Bloch-Kato Selmer group

$$H_f^1(V) := \text{Ker} \left(H^1(G_{F,S}, V) \rightarrow \prod_{v \in S} \frac{H^1(D_v, V)}{L_v} \right).$$

We make the following additional hypotheses

C1) $H_f^1(V) = H_f^1(V^*(1)) = 0$,

C2) $H^0(G_{F,S}, V) = H^0(G_{F,S}, V^*(1)) = 0$,

C3) φ on $\mathbf{D}_{\text{st}}(V|_{F_p})$ is semisimple at $1 \in F_p^{\text{ur}} \otimes_{\mathbb{Q}_p} E$ and $p^{-1} \in F_p^{\text{ur}} \otimes_{\mathbb{Q}_p} E$ for all $\mathfrak{p} \mid p$,

C4) $\mathbf{D}_{\text{rig}}^\dagger(V|_{F_p})$ has no saturated sub-quotient of type $U_{m,k}$ for all $\mathfrak{p} \mid p$.

Note that if V satisfies the previous four conditions, so does $V^*(1)$.

The first two conditions tell us that the Poitou-Tate sequence reduces to

$$H^1(G_{F,S}, V) \cong \bigoplus_{v \in S} \frac{H^1(D_v, V)}{H_f^1(V, F_v)}. \quad (3.1)$$

For each $\mathfrak{p} \mid p$ we denote by $V_{\mathfrak{p}}$ the restriction to G_{F_p} of V . We choose a regular sub-module $D_{\mathfrak{p}} \subset \mathbf{D}_{\text{st}}(V_{\mathfrak{p}})$ and define a filtration $(D_{\mathfrak{p},i})$ of $\mathbf{D}_{\text{st}}(V_{\mathfrak{p}})$.

$$D_{\mathfrak{p},i} = \begin{cases} 0 & i = -2, \\ (1 - p^{-1}\varphi)D_{\mathfrak{p}} + N(D_{\mathfrak{p}}^{\varphi=1}) & i = -1, \\ D_{\mathfrak{p}} & i = 0, \\ D_{\mathfrak{p}} + \mathbf{D}_{\text{st}}(V_{\mathfrak{p}})^{\varphi=1} \cap N^{-1}(D_{\mathfrak{p}}^{\varphi=p^{-1}}) & i = 1, \\ \mathbf{D}_{\text{st}}(V_{\mathfrak{p}}) & i = 2. \end{cases} \quad (3.2)$$

We have that $D_{\mathfrak{p},1}/D_{\mathfrak{p},-1}$ coincides with the eigenvectors of φ on $\mathbf{D}_{\text{st}}(V_{\mathfrak{p}})$ of eigenvalue 1 resp. p^{-1} and which are in the kernel of N resp. in the image of N .

This filtration induces a filtration on $\mathbf{D}_{\text{rig}}^\dagger(V_{\mathfrak{p}})$. Namely, we pose

$$F_i \mathbf{D}_{\text{rig}}^\dagger(V_{\mathfrak{p}}) = \mathbf{D}_{\text{rig}}^\dagger(V_{\mathfrak{p}}) \cap (D_{\mathfrak{p},i} \otimes \mathcal{R}_{F_p, \log}[t^{-1}]).$$

We define

$$W_{\mathfrak{p}} := F_1 \mathbf{D}_{\text{rig}}^{\dagger}(V_{\mathfrak{p}}) / F_{-1} \mathbf{D}_{\text{rig}}^{\dagger}(V_{\mathfrak{p}}).$$

We have

$$W_{\mathfrak{p}} = W_{\mathfrak{p},0} \bigoplus W_{\mathfrak{p},1} \bigoplus M_{\mathfrak{p}}$$

where $t_{\mathfrak{p},0} = \dim_E H^0(W) = \text{rank}_{\mathcal{R}_{F_{\mathfrak{p}}}} W_0$, $t_{\mathfrak{p},1} = \dim_E H^0(W^*(1)) = \text{rank}_{\mathcal{R}_{F_{\mathfrak{p}}}} W_1$ and M sits in a non split sequence

$$0 \rightarrow M_{\mathfrak{p},0} \xrightarrow{f} M_{\mathfrak{p}} \xrightarrow{g} M_{\mathfrak{p},1} \rightarrow 0$$

such that $\text{gr}^0(\mathbf{D}_{\text{rig}}^{\dagger}(V_{\mathfrak{p}})) = W_{\mathfrak{p},0} \oplus M_{\mathfrak{p},0}$ and $\text{gr}^1(\mathbf{D}_{\text{rig}}^{\dagger}(V_{\mathfrak{p}})) = W_{\mathfrak{p},1} \oplus M_{\mathfrak{p},1}$.

We can prove exactly in the same way as [Ben11, Proposition 2.1.7 (i)] that **C4** implies $\text{rank}_{\mathcal{R}_{F_{\mathfrak{p}}}} M_1 = \text{rank}_{\mathcal{R}_{F_{\mathfrak{p}}}} M_0$.

In order to define the \mathcal{L} -invariant we shall follow verbatim Benois' construction. For sake of notation, we write $\mathbf{D}_{\mathfrak{p}}^{\dagger}$ for $\mathbf{D}_{\text{rig}}^{\dagger}(V_{\mathfrak{p}})$. We obtain from [Ben11, Proposition 1.4.4 (i)]

$$H_{\mathfrak{f}}^1(\text{gr}^2(\mathbf{D}_{\mathfrak{p}}^{\dagger})) = H^0(\text{gr}^2(\mathbf{D}_{\mathfrak{p}}^{\dagger})) = 0.$$

We deduce the following isomorphism

$$H_{\mathfrak{f}}^1(F_1 \mathbf{D}_{\mathfrak{p}}^{\dagger}) = H_{\mathfrak{f}}^1(\mathbf{D}_{\mathfrak{p}}^{\dagger}) = H_{\mathfrak{f}}^1(F_{\mathfrak{p}}, V). \quad (3.3)$$

As the Hodge-Tate weights of $F_{-1} \mathbf{D}_{\mathfrak{p}}^{\dagger}$ are < 0 , we obtain from [Ben11, Proposition 1.5.3 (i)] and Poiteau-Tate duality $H^2(F_{-1} \mathbf{D}_{\mathfrak{p}}^{\dagger}) = 0$. Using the long exact sequence associated to

$$0 \rightarrow F_{-1} \mathbf{D}_{\mathfrak{p}}^{\dagger} \rightarrow F_1 \mathbf{D}_{\mathfrak{p}}^{\dagger} \rightarrow W_{\mathfrak{p}} \rightarrow 0$$

we see that

$$\frac{H^1(W_{\mathfrak{p}})}{H_{\mathfrak{f}}^1(W_{\mathfrak{p}})} = \frac{H^1(F_{-1} \mathbf{D}_{\mathfrak{p}}^{\dagger})}{H_{\mathfrak{f}}^1(F_{\mathfrak{p}}, V)}.$$

We suppose now

C5) $W_{\mathfrak{p},0} = 0$ for all $\mathfrak{p} \mid p$.

Write $\text{gr}^1(\mathbf{D}_{\mathfrak{p}}^{\dagger}) = \bigoplus_{i=1}^{t_{\mathfrak{p},1}+r_{\mathfrak{p}}} \mathcal{R}_{F_{\mathfrak{p}}}(\prod_{\tau_{\mathfrak{p}}} \tau_{\mathfrak{p}}(z)^{m_{i,\tau_{\mathfrak{p}}}})$. We define the $2(t_{\mathfrak{p},1} + r_{\mathfrak{p}})$ -dimensional subspace obtained as the image of

$$\text{Ind}_{\mathfrak{p}} := \text{Im} \left(H^1 \left(\bigoplus_{i=1}^{t_{\mathfrak{p},1}+r_{\mathfrak{p}}} \mathcal{R}_{\mathbb{Q}_p} \left(z^{\sum_{\tau_{\mathfrak{p}}} m_{i,\tau_{\mathfrak{p}}}} \right) \right) \right) \subset H^1(\text{gr}^1(\mathbf{D}_{\mathfrak{p}}^{\dagger})). \quad (3.4)$$

We define

$$T_{\mathfrak{p}} = (H^1(F_1 \mathbf{D}_{\mathfrak{p}}^{\dagger}) \cap \text{Ind}_{\mathfrak{p}}) / H_{\mathfrak{f}}^1(F_{\mathfrak{p}}, V).$$

It has dimension $t_{\mathfrak{p},1} + r_{\mathfrak{p}}$.

Write $t = \sum_{\mathfrak{p}} t_{\mathfrak{p},1} + r_{\mathfrak{p}}$. We have a unique t -dimensional subspace $H^1(D, V)$ of $H^1(G_{F,S}, V)$ projecting via 3.1 to $\bigoplus_{\mathfrak{p}} T_{\mathfrak{p}}$. We have an isomorphism [Ben11, Proposition 1.5.9]

$$H^1(\bigoplus_{i=1}^{t_{\mathfrak{p},1}+r_{\mathfrak{p}}} \mathcal{R}_{\mathbb{Q}_p}(z^{\sum_{\tau_{\mathfrak{p}}} m_{i,\tau_{\mathfrak{p}}}})) = \mathcal{D}_{\text{cris}}(W_1 \oplus M_1) \oplus \mathcal{D}_{\text{cris}}(W_1 \oplus M_1)$$

We shall denote the two projections by ι_f and ι_c .

A canonical basis is given by the above mentioned cocycles x_m (resp. y_m) defined in (resp. right after) Proposition 2.5.

By abuse of notation, we still denote by ι_f resp. ι_c be the projection of $H^1(D, V)$ to $\mathcal{D}_{\text{cris}}(W)$ via ι_f resp. ι_c . By the remark after Lemma 2.11 and the definition of $T_{\mathfrak{p}}$, we have that $H^1(D, V)$ surjects into $\mathcal{D}_{\text{cris}}$ via ρ_c . Summing up, we can give the following definition;

Definition 3.5. *The \mathcal{L} -invariant of the pair (V, D) is*

$$\mathcal{L}(D, V) := \det(\iota_f \circ \iota_c^{-1}),$$

where the determinant is calculated w.r.t. the basis $(x_{m_i}, y_{m_j})_{1 \leq i, j \leq t}$.

Remark 3.6. *There is no a priori reason for which $\mathcal{L}(D, V)$ should be non-zero.*

In the case $W_{\mathfrak{p}} = M_{\mathfrak{p}}$ we see from the description of $H^1(F_1 \mathbf{D}_{\mathfrak{p}}^{\dagger})$ that the space $T_{\mathfrak{p}}$ depends only on $V|_{F_{\mathfrak{p}}}$ exactly as in the classical case.

3.2 Comparison with Benois' definition

Fix a global field F and let $\{\mathfrak{p}\}$ be the set of primes above p .

Let G_p denote a fixed decomposition group at p in $G_{\mathbb{Q}}$ and let \mathfrak{p}_0 be the corresponding place of F . Let $G_{\mathfrak{p}_0, F}$ be the decomposition group at \mathfrak{p}_0 in G_F . For each other place \mathfrak{p} above p in F , we have $G_{\mathfrak{p}} = \sigma_{\mathfrak{p}} G_{\mathfrak{p}_0, F} \sigma_{\mathfrak{p}}^{-1}$. We shall denote by $G_{\mathfrak{p}, F}$ the corresponding decomposition group in G_F . Consider a p -adic Galois representation

$$V : G_F \rightarrow \text{GL}_n(E).$$

We shall suppose E big enough to contain the Galois closure of $F_{\mathfrak{p}}$, for all \mathfrak{p} . As before, we suppose V semistable at all primes above p . We have then

$$\text{Ind}_F^{\mathbb{Q}}(V) \cong_{G_p} \bigoplus_{\mathfrak{p}} \sigma_{\mathfrak{p}}^{-1} \text{Ind}_{G_{\mathfrak{p}, F}}^{G_{\mathfrak{p}}} V|_{G_{\mathfrak{p}, F}}$$

where $\sigma_{\mathfrak{p}} \in G_p \setminus \text{Hom}(F, \overline{\mathbb{Q}})$.

Consider the (φ, Γ) -module

$$\mathbf{D}^{\dagger} := \mathbf{D}_{\text{rig}}^{\dagger} \left(\text{Ind}_F^{\mathbb{Q}} V \right).$$

We let D be the regular (φ, N) -module of $\mathcal{D}_{\text{st}}(\mathbf{D}^{\dagger})$ induced by $\{D_{\mathfrak{p}}\}_{\mathfrak{p}}$. As before we have a filtration $(F_i \mathbf{D}^{\dagger})$ on \mathbf{D}^{\dagger} induced by the filtration on D . We denote by W the quotient $F_1 \mathbf{D}^{\dagger} / F_{-1} \mathbf{D}^{\dagger}$. Note that it is semistable. We write $W = W_0 \oplus M \oplus W_1$. We suppose **C1-C5** of the previous section.

Lemma 3.7. *Let M be as in (2.9). We have*

$$0 \rightarrow \text{Ind}(M_0) \rightarrow \text{Ind}(M) \rightarrow \text{Ind}(M_1) \rightarrow 0.$$

We can now compare our definition of \mathcal{L} -invariant with Benois'.

Proposition 3.8. *We have a commutative diagram*

$$\begin{array}{ccccc} H^1(G_{\mathbb{Q}, S}, \text{Ind}(V)) & \longleftarrow & H^1(\text{Ind}(D), \text{Ind}(V)) & \xrightarrow{\text{Res}_p} & \frac{H^1(F_1 \mathbf{D}^{\dagger}(\text{Ind}(V)))}{H_{\mathfrak{f}}^1(G_p, \text{Ind}(V))} = \frac{H^1(F_{-1} \mathbf{D}^{\dagger})}{H_{\mathfrak{f}}^1(G_p, \text{Ind}(V))}. \\ \downarrow & & \downarrow & & \downarrow \iota_p \\ H^1(G_{F, S}, V) & \longleftarrow & H^1(D, V) & \xrightarrow{\oplus_{\mathfrak{p}} \text{Res}_{\mathfrak{p}}} & \prod_{\mathfrak{p}} T_{\mathfrak{p}} \end{array}$$

whose vertical arrows are isomorphism.

Proof. We follow [Hid06, §3.4.4]. Recall that we wrote $\mathbf{D}_{\mathfrak{p}}^{\dagger}$ for $\mathbf{D}_{\text{rig}}^{\dagger}(V_{\mathfrak{p}})$. Shapiro's lemma tells us that

$$\frac{H^1(G_p, \text{Ind}_F^{\mathbb{Q}} V)}{H_f^1(G_p, \text{Ind}_F^{\mathbb{Q}} V)} \stackrel{\iota_p}{\cong} \bigoplus_{\mathfrak{p}} \frac{H^1(\mathbf{D}_{\mathfrak{p}}^{\dagger})}{H_f^1(\mathbf{D}_{\mathfrak{p}}^{\dagger})}.$$

We are left to show that $H^1(F_1 \mathbf{D}^{\dagger}(\text{Ind}(V)))$ is sent by $\text{Res}_{\mathfrak{p}}$ into $(H^1(F_1 \mathbf{D}_{\mathfrak{p}}^{\dagger}) \cap \text{Inv}_{\mathfrak{p}})$ and we shall conclude by dimension counting.

We have then an injection

$$F_1 \mathbf{D}^{\dagger}(\text{Ind}(V)) \hookrightarrow \bigoplus_{\mathfrak{p}} \text{Ind}(F_1(\mathbf{D}_{\text{rig}}^{\dagger}(V_{\mathfrak{p}}))).$$

Then clearly the image of ι_p lands in $H^1(F_1 \mathbf{D}_{\mathfrak{p}}^{\dagger})$. But we have also the injection

$$\text{gr}^1(\mathbf{D}_{\text{rig}}^{\dagger}(\text{Ind} V)) \hookrightarrow \bigoplus_{\mathfrak{p}} \text{Ind}(\text{gr}^1(\mathbf{D}_{\text{rig}}^{\dagger}(V_{\mathfrak{p}})))$$

induced by (2.7). Then the image of ι_p lands in $\text{Inv}_{\mathfrak{p}}$ and we are done. \square

Proposition 3.9. *We have $\mathcal{L}(V) = \mathcal{L}(\text{Ind}_F^{\mathbb{Q}}(V))$.*

Proof. After the previous proposition, what we have to do is to notice that the cocycle $x_{\sum m_{\tau}}$ (resp. $y_{\sum m_{\tau}}$) is identified with x_m (resp. y_m). \square

4 Siegel-Hilbert modular forms, the local case

The calculation of the \mathcal{L} -invariant requires to produce explicit cocycles in $H^1(D, V)$; when V appears in $\text{Ad}(V')$ for a certain representation V' we can sometimes use the method of Mazur and Tilouine [MT90] to produce these cocycles. This has been done in many case for the symmetric square [Hid04, Mok12] and generalized to symmetric powers of the Galois representation associated with Hilbert modular forms in [Hid07, HJ13]. The main limit of this approach is that for most of the representations V is this computationally heavy to make it appear as the quotient of an adjoint representations.

In the case $\mathbf{D}_{\text{rig}}^{\dagger}(V) = W = M$ the situation is way simpler; if $t = 1$ it has been proved in [Ben10] that to produce the cocycle in $H^1(V, D)$ it is enough to find deformations of $V|_{\mathbb{Q}_p}$.

We shall generalize the construction of Benois to our situation in the case $W_{\mathfrak{p}} = M_{\mathfrak{p}}$ and $r_{\mathfrak{p}} = 1$. This will allow us to give a complete formula for the \mathcal{L} -invariant of the Galois representations associated with a Siegel-Hilbert modular form which is Steinberg at all primes above p .

4.1 The case $t_{\mathfrak{p}} = r_{\mathfrak{p}} = 1$

We suppose now that $W_{\mathfrak{p}} = M_{\mathfrak{p}}$ and $r_{\mathfrak{p}} = 1$. For sake of notation, in this section we shall drop the index \mathfrak{p} .

All that we have to do is to check that the calculation of [Ben11, Theorem 2] works in our setting.

We write as before

$$0 \rightarrow M_0 \rightarrow M \rightarrow M_1 \rightarrow 0$$

and, only in this subsection, we shall write δ for the character defining M_0 and ψ for the character defining M_1 . We have $\delta(z) = |N_{L/\mathbb{Q}_p}(z)|_p \prod_{\tau} \tau(z)^{k_{\tau}}$ with $k_{\tau} \geq 1$ and $\psi(z) = \prod_{\tau} \tau(z)^{m_{\tau}}$ with $m_{\tau} \leq 0$. We consider an infinitesimal deformation

$$0 \rightarrow M_{0,A} \rightarrow M_A \rightarrow M_{1,A} \rightarrow 0,$$

over $A = E[T]/(T^2)$. We suppose that $M_{0,A}$ (resp. $M_{1,A}$) is an infinitesimal deformation of M_0 (resp. M_1). We shall write δ_A and ψ_A for the corresponding one-dimensional character.

Theorem 4.1. *Suppose that $\mathrm{d} \log(\delta_A \psi_A^{-1})(\chi_{\mathrm{cycl}}(\gamma_{\mathbb{Q}_p}))|_{T=0} \neq 0$; then*

$$\mathcal{L}(M, M_0) = -\log(\chi_{\mathrm{cycl}}(\gamma_{\mathbb{Q}_p})) \frac{\mathrm{d} \log(\delta_A \psi_A^{-1})(p)|_{T=0}}{\mathrm{d} \log(\delta_A \psi_A^{-1})(\chi_{\mathrm{cycl}}(\gamma_{\mathbb{Q}_p}))|_{T=0}}$$

Proof. Recall the definition of Ind in (3.4). We have a vector $v = ax_m + by_m$ in $H^1(F_1 \mathbf{D}^\dagger) \cap \mathrm{Ind}$. By definition $\mathcal{L}(M) = ab^{-1}$. The extension $M_{j,A}$ provides us with connecting morphisms $B_j^i : H^i(M_j) \rightarrow H^{i+1}(M_j)$. We have by definition

$$B_1^0(t^{-m}e_m) = \mathrm{cl}(\mathrm{d} \log(\delta_A)(\pi_L)t^{-m}e_m, \mathrm{d} \log(\delta_A)(\chi_{\mathrm{cycl}}(\gamma))t^{-m}e_m) \quad (4.2)$$

$$= \mathrm{d} \log(\delta_A)(\pi_L)x_m + \mathrm{d} \log(\delta_A)(\chi_{\mathrm{cycl}}(\gamma))y_m. \quad (4.3)$$

As in [Ben10, §3.2] we consider the dual extension

$$0 \rightarrow M_1^*(\chi_{\mathrm{cycl}}) \rightarrow M^*(\chi_{\mathrm{cycl}}) \rightarrow M_1^*(\chi_{\mathrm{cycl}}) \rightarrow 0,$$

and we shall denote with a * the corresponding map in the long exact sequence of cohomology. We have hence $\ker(\Delta_1) \perp \mathrm{Im}(\Delta_0^*)$ under duality, and a map

$$H^1(M_1^*) \rightarrow H^1(\mathcal{R}_{\mathbb{Q}_p}(|z|z^{1-\sum_\tau m_\tau})).$$

By duality again, we deduce that the image of Δ_0^* inside the target of the above arrow is

$$a\alpha_{1-\sum_\tau m_\tau} + b\beta_{1-\sum_\tau m_\tau},$$

where $\alpha_{1-\sum_\tau m_\tau}$ (resp. $\beta_{1-\sum_\tau m_\tau}$) is the dual of $x_{\sum_\tau m_\tau}$ (resp. $y_{\sum_\tau m_\tau}$).

We consider now the map

$$B_1^{1*} : H^1(M_1^*(\chi_{\mathrm{cycl}})) \rightarrow H^2(M_1^*(\chi_{\mathrm{cycl}})) = H^2(\mathcal{R}_{\mathbb{Q}_p}(|z|z^{1-\sum_\tau m_\tau})) \cong E,$$

where the identity is the dual of the identity induced by (2.7).

We can use [Ben10, Proposition 2.4] to see that after the above identification of H^2 with E we have

$$B_1^{1*}(\alpha_{1-\sum_\tau m_\tau}) = \mathrm{clog}_p(\chi_{\mathrm{cycl}}(\gamma_{\mathbb{Q}_p}))^{-1} \mathrm{d} \log_p(\delta_A)(\chi_{\mathrm{cycl}}(\gamma_{\mathbb{Q}_p}))|_{T=0}, \quad (4.4)$$

$$B_1^{1*}(\beta_{1-\sum_\tau m_\tau}) = c \mathrm{d} \log_p(\delta_A)(p)|_{T=0}, \quad (4.5)$$

where $c \in E^\times$. We consider the following anti-commutative diagram

$$\begin{array}{ccc} H^0(M_0^*(\chi_{\mathrm{cycl}})) & \xrightarrow{\Delta_0^*} & H^1(M_1^*(\chi_{\mathrm{cycl}})) \\ \downarrow B_0^{1*} & & \downarrow B_1^{1*} \\ H^1(M_0^*(\chi_{\mathrm{cycl}})) & \xrightarrow{\Delta_1^*} & H^2(M_1^*(\chi_{\mathrm{cycl}})) \end{array}$$

which implies

$$B_1^{1*} \Delta_0^* = -\Delta_1^* B_0^{1*}.$$

We calculate this identity on t^{1-k} . Applying (4.2) and (4.3) to $\psi_A^{-1} \chi_{\mathrm{cycl}}|_{\mathbb{Q}_p^\times}$, (4.2) and (4.3) to $\delta_A^{-1} \chi_{\mathrm{cycl}}|_{\mathbb{Q}_p^\times}$ and using [Ben10, (3.6)] which says

$$\Delta_1^* B_0^{1*}(t^{1-k}) = c(b \log_p(\delta_A)(p) + a \mathrm{d} \log_p(\delta_A)(\chi_{\mathrm{cycl}}(\gamma)))$$

we get

$$b^{-1}a = -\log_p(\chi_{\text{cycl}}(\gamma_{\mathbb{Q}_p})) \frac{d \log_p(\delta_A \psi_A^{-1})(p)|_{T=0}}{d \log(\delta_A \psi_A^{-1})(\chi_{\text{cycl}}(\gamma_{\mathbb{Q}_p}))|_{T=0}}.$$

□

Remark 4.6. *In particular, this theorem proves that this definition of \mathcal{L} -invariant is compatible with the Coleman or Fontaine-Mazur ones [Pot14, Zha14].*

4.2 Calculation of the \mathcal{L} -invariant for Steinberg forms

We fix a totally real field F . Let I be the set of real embeddings. Fix two embeddings

$$\mathbb{C}_p \hookleftarrow \overline{\mathbb{Q}} \hookrightarrow \mathbb{C}$$

as before. We partition $I = \sqcup_{\mathfrak{p}} I_{\mathfrak{p}}$ according to the p -adic place which each embedding induces. We shall denote by $q_{\mathfrak{p}}$ the residual cardinality for each prime ideal \mathfrak{p} . We consider an irreducible representation π of $\text{GSp}_{2g/F}$ algebraic of weight $k = (k_{\tau})_{\tau}$, where $(k_{\tau}) = (k_{\tau,1}, \dots, k_{\tau,g}; k_0)$ (k_0 is a parallel weight for $\text{Res}_{\overline{F}}^{\mathbb{Q}}(\mathbb{G}_m)$). With $k_{\tau,1} \leq k_{\tau,2} \leq \dots \leq k_{\tau,g}$. If $k_{\tau,1} \geq g+1$, then the weight is cohomological. The cohomological weight of π is then

$$(\mu_{\tau})_{\tau} = (k_{\tau})_{\tau} - (g+1, \dots, g+1; 0)_{\tau}.$$

For parallel weight k , we shall choose $k_0 = gk$.

We describe now the conjectural Galois representation associated with π . We have a spin Galois representation V_{spin} (whose image is contained in GL_{2g}) and a standard Galois representation V_{sta} (whose image is contained in GL_{2g+1}) given respectively by the spinorial and the standard representation of $\text{GSpin}_{2g+1} = {}^L\text{GSp}_{2g}$.

Thanks to the work of Scholze [Sch13] we dispose now of the standard Galois representation (see for example [HJ13, Theorem 18]). We also know the existence of the spin representation in many cases [KS14].

We recall now some expected properties of these Galois representations. Our main reference is [HJ13, §3.3]. We will make the following assumption on π at p ;

for each $\mathfrak{p} \mid p$ either $\pi_{\mathfrak{p}}$ is spherical or Steinberg.

We explain what we mean by Steinberg. Consider the Satake parameters at \mathfrak{p} , normalized as in [BS00, Corollary 3.2], $(\alpha_{\mathfrak{p},1}, \dots, \alpha_{\mathfrak{p},g})$. We have the following theorem on Iwahori spherical representation of $\text{GSp}_{2g}(F_{\mathfrak{p}})$ [Tad94, Theorem 7.9]

Theorem 4.7. *Let $\alpha_1, \dots, \alpha_g, \alpha$ be $g+1$ character of $F_{\mathfrak{p}}^{\times}$. Let $B_{\text{GSp}_{2g}}$ be the Borel subgroup of $\text{Sp}_{2g}(F_{\mathfrak{p}})$. Then $\text{Ind}_{B_{\text{GSp}_{2g}}}^{\text{GSp}_{2g}(F_{\mathfrak{p}})}(\alpha_1 \times \dots \times \alpha_g \rtimes \alpha)$ is not irreducible if and only if one of the following conditions is satisfied:*

- i) *There exist at least three indexes i such that α_i has exact order two and the α_i 's are mutually distinct;*
- ii) *There exists i such that $\alpha_i = |\text{N}(\)|_{\mathfrak{p}}^{\pm 1}$;*
- iii) *There exist i and j such that $\alpha_i = |\text{N}(\)|_{\mathfrak{p}}^{\pm 1} \alpha_j^{\pm 1}$.*

Remark 4.8. *As shown in [HJ13, Lemma 19], such α points are contained in a proper subset of the Hecke eigenvariety for GSp_{2g} .*

Definition 4.9. We say that $\pi_{\mathfrak{p}}$ is Steinberg if $\alpha_i = |N(\)|_{\mathfrak{p}}^{-1} \alpha_1$.

If $\pi_{\mathfrak{p}}$ is Steinberg at p , then $\alpha_{\mathfrak{p},i}(\varpi_{\mathfrak{p}}) = q_{\mathfrak{p}}^i \alpha_{\mathfrak{p},1}(\varpi_{\mathfrak{p}})$.

Trivial zeros appears also for automorphic forms which are only partially Steinberg at \mathfrak{p} and can be dealt exactly at the same way as the parallel one but for the sake of notation we prefer not to deal with them.

To each $g+1$ non-zero elements $(t_1, \dots, t_g; t_0) \in (A^\times)^{g+1}$ we associate the diagonal matrix

$$u(t_1, \dots, t_g; t_0) := (t_1, \dots, t_g, t_0 t_g^{-1}, \dots, t_0 t_1^{-1})$$

of $\mathrm{GSp}_{2g}(A)$.

For $1 \leq i \leq g-1$ we denote by $u_{\mathfrak{p},i}$ the diagonal matrix associated with $(1, \dots, 1, \varpi_{\mathfrak{p}}^{-1}, \dots, \varpi_{\mathfrak{p}}^{-1}; \varpi_{\mathfrak{p}}^{-2})$, where $\varpi_{\mathfrak{p}}$ appears i times; we also denote by $u_{\mathfrak{p},0}$ the diagonal matrix corresponding to $(1, \dots, 1; \varpi_{\mathfrak{p}}^{-1})$.

Definition 4.10. The Hecke operators $U_{\mathfrak{p},i}$, for $1 \leq i \leq g$ are defined as the double coset operator $[\mathrm{Iw} u_{\mathfrak{p},g-i} \mathrm{Iw}]$.

We have that $U_{\mathfrak{p},g}$ is the “classical” U_p operator [BS00, §0]. We shall say then that π is of finite slope for $U_{\mathfrak{p},g}$ if $U_{\mathfrak{p},g}$ has eigenvalue $\alpha_{\mathfrak{p},0} \neq 0$ on $\pi_{\mathfrak{p}}$.

We are interested to study the possible p -stabilization of π (i.e. Iwahori fixed vectors). If $\pi_{\mathfrak{p}}$ is unramified at \mathfrak{p} , we have then $2^g g!$ choices (see [HJ13, Lemma 16] or [BS00, Proposition 9.1]). If $\pi_{\mathfrak{p}}$ is Steinberg, we have instead only one possible choice, as the monodromy N has maximal rank.

Suppose that we can lift π to an automorphic representation $\pi^{(2^g)}$ of GL_{2g} . We suppose also that we can lift π to an automorphic representation $\pi^{(2^{g+1})}$ of GL_{2g+1} .

Let $V = V_{\mathrm{spin}}$ (resp. V_{sta}) be the Galois representation associated with $\pi^{(2^g)}$ (resp. $\pi^{(2^{g+1})}$). We make the following assumption

LGp) V is semistable at all $\mathfrak{p} \mid p$ and strong local-global compatibility at $l = p$ holds.

These hypotheses are conjectured to be always true for f as above. Arthur’s transfer from GSp_{2g} to GL_{2g+1} has been proven in [Xu] (note that it is now unconditional [MW]) and for $V = V_{\mathrm{sta}}$ this hypothesis is then verified thanks to [Car13, Theorem 1.1]. These hypotheses are also satisfied in many cases for $V = V_{\mathrm{spin}}$ in genus 2 (see [AS06, PSS14]).

Roughly speaking, we require that

$$\mathrm{WD}(V|_{F_{\mathfrak{p}}})^{\mathrm{ss}} \cong \iota_n^{-1} \pi_{\mathfrak{p}}^{(n)},$$

where $\mathrm{WD}(V|_{F_{\mathfrak{p}}})$ is the Weil-Deligne representation associated with $V|_{F_{\mathfrak{p}}}$ à la Berger, $\pi_{\mathfrak{p}}^{(n)}$ is the component at \mathfrak{p} of $\pi^{(n)}$, and ι_n is the local Langlands correspondence for $\mathrm{GL}_n(F_{\mathfrak{p}})$ geometrically normalized ($n = 2g+1$ when V is the standard representation and $n = 2^g$ when V is the spinorial representation).

When $\pi_{\mathfrak{p}}$ is an irreducible quotient of $\mathrm{Ind}_B^{\mathrm{GSp}_{2g}}(\alpha_{\mathfrak{p},1} \otimes \dots \otimes \alpha_{\mathfrak{p},g})$ we have that the Frobenius eigenvalues on $\mathrm{WD}(V_{\mathrm{spin}|_{F_{\mathfrak{p}}}})^{\mathrm{ss}}$ are the 2^g numbers

$$\left(\begin{array}{c} \alpha_{\mathfrak{p},0} \\ \prod_{\substack{0 \leq r \leq g \\ 1 \leq i_1 < \dots < i_r \leq g}} \alpha_{\mathfrak{p},i_1}(\varpi_{\mathfrak{p}}) \cdots \alpha_{\mathfrak{p},i_r}(\varpi_{\mathfrak{p}}) \end{array} \right).$$

The ones on $\mathrm{WD}(V_{\mathrm{sta}|F_p})^{\mathrm{ss}}$ are

$$(\alpha_{p,g}^{-1}(\varpi_p), \dots, \alpha_{p,1}^{-1}(\varpi_p), 1, \alpha_{p,1}(\varpi_p), \dots, \alpha_{p,g}(\varpi_p)).$$

Moreover, the monodromy operator should have maximal rank (i.e. one-dimensional kernel) if we are Steinberg or be trivial otherwise. (This is also a consequence of the weight-monodromy conjecture for V .)

Let p be a p -adic place of V and let τ be a complex place in I_p . The Hodge-Tate weights of $V_{\mathrm{spin}|F_p}$ at τ are then

$$\left(\frac{k_0}{2} + \frac{1}{2} \sum_{i=1}^g \varepsilon(i)(k_{\tau,i} - i) \right)_{\varepsilon},$$

where ε ranges among the 2^g maps from $\{1, \dots, g\}$ to $\{\pm 1\}$.

The one of $V_{\mathrm{sta}|F_p}$ are $(1 - k_{\tau,g}, \dots, g - k_{\tau,1}, 0, k_{\tau,1} - g, \dots, k_{\tau,g} - 1)$.

Thanks to work of Tilouine-Urban [TU99], Urban [Urb11], Andreatta-Iovita-Pilloni [AIP12] we have families of Siegel modular forms;

Theorem 4.11. *Let $\mathcal{W} = \mathrm{Hom}_{\mathrm{cont}} \left(\mathbb{Z}_p^\times \times ((\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times)^g, \mathbb{C}_p^\times \right)$ be the weight space. There exist an affinoid neighborhood \mathcal{U} of $\kappa_0 = ((z, (z_i)_{i=1}^g) \mapsto z^{k_0} \prod_{\tau \in I} \prod_i \tau(z_i)^{k_{\tau,i}})$ in \mathcal{W} , an equidimensional rigid variety $\mathcal{X} = \mathcal{X}_\pi$ of dimension $dg + 1$, a finite surjective map $w : \mathcal{X} \rightarrow U$, a character $\Theta : \mathcal{H}^{N_p} \rightarrow \mathcal{O}(\mathcal{X})$, and a point x in \mathcal{X} above \underline{k} such that $x \circ \Theta$ corresponds to the Hecke eigensystem of π .*

Moreover, there exists a dense set of points x of \mathcal{X} coming from classical cuspidal Siegel-Hilbert automorphic forms of weight $(k_{\tau,i}; k_0)$ which are regular and spherical at p .

Remark 4.12. *Assuming Leopoldt conjecture, the multiplicative group appearing in the definition of \mathcal{W} is, up to a finite subgroup, $((\mathcal{O}_F \otimes_{\mathbb{Z}} \mathbb{Z}_p)^\times)^{g+1} / \overline{\mathcal{O}_F^\times}$ (i.e. the \mathbb{Z}_p -points of the torus of $\mathrm{Res}_F^{\mathbb{Q}}(\mathrm{GSp}_{2g})$ modulo the \mathbb{Z}_p -points of the center).*

This allows us to define two pseudo-representations $R_? : G_{\mathbb{Q}} \rightarrow \mathcal{O}(\mathcal{X})$, for $? = \mathrm{spin}, \mathrm{sta}$, interpolating the trace of the representations associated with classical Siegel forms [BC09, Proposition 7.5.4]. Suppose now that $V_?$ is absolutely irreducible (this is conjectured to hold when π is Steinberg at least at one prime); we have then, shrinking \mathcal{U} around \underline{k} if necessary, a big Galois representation $\rho_?$ with value in $\mathrm{GL}_n(\mathcal{O}(\mathcal{X}))$ such that $\mathrm{Tr}(\rho_?) = R_?$ [BC09, page 214].

For $1 \leq j < g$ we define $\lambda_p(u_{p,g-j}) = \varpi_p^{\sum_{\tau \in I_p} k_{\tau,1} + \dots + k_{\tau,j} - k_0}$ and $\lambda_p(u_{p,0}) = \varpi_p^{\sum_{\tau \in I_p} (k_{\tau,1} + \dots + k_{\tau,g} - k_0)/2}$. We have analytic functions $\beta_{p,j} := \Theta(U_{p,j} | \lambda_p(u_{p,g-j}) |_p) \in \mathcal{O}(\mathcal{X})$. We proceed now as in [HJ13]. We recall the following theorem [Liu13, Theorem 0.3.4];

Theorem 4.13. *Let $\rho : G_{F_p} \rightarrow \mathrm{GL}_n(\mathcal{O}(\mathcal{X}))$ be a continuous representation. Suppose that there exist $\kappa_1(x), \dots, \kappa_n(x)$ in $F_p \otimes_{\mathbb{Q}_p} \mathcal{O}(\mathcal{X})$, $F_1(x), \dots, F_d(x)$ in $\mathcal{O}(\mathcal{X})$, and a Zariski dense set of points $Z \subset \mathcal{X}$ such that*

- *for any x in \mathcal{X} , the Hodge-Tate weights of ρ_x are $\kappa_1(x), \dots, \kappa_n(x)$;*
- *for any z in Z , ρ_z is crystalline;*
- *for any z in Z , $\kappa_{\tau,1}(z) < \dots < \kappa_{\tau,n}(z)$, for all $\tau \in I_p$;*
- *for any z in Z , the eigenvalues of φ^{f_p} on $\mathcal{D}_{\mathrm{cris}}(V_z)$ are $\prod_{\tau \in I_p} \tau(\varpi_p)^{\kappa_{\tau,1}(z)} F_1(z), \dots, \prod_{\tau \in I_p} \tau(\varpi_p)^{\kappa_{\tau,n}(z)} F_n(z)$;*
- *for any C in \mathbb{R} , defines $Z_C \subset Z$ as the set of points z such that for all $I, J \subset \{1, \dots, n\}$ such that $|\sum_{i \in I} \kappa_{\tau,i}(z) - \sum_{j \in J} \kappa_{\tau,j}(z)| > C$ for all $\tau \in I_p$. We require that for all $z \in Z$ and $C \in \mathbb{R}$, Z_C accumulates at z .*

- for $1 \leq i \leq n$ there exist $\chi_i : \mathcal{O}_{F_p}^\times \rightarrow \mathcal{O}(\mathcal{X})^\times$ such that $\chi_i(u) = \prod_\tau \tau(u)^{\kappa_{\tau,i}(x)}$.

Then, for all x in \mathcal{X} non-critical and regular ($\kappa_1(x) < \dots < \kappa_n(x)$ and the eigenvalues of φ on $\bigwedge^i \mathcal{D}_{\text{cris}}(V_x)$ are distinct for all i) there exists a neighborhood U of x such that ρ_U is trianguline and its graded pieces are $\mathcal{R}_U(\chi_i)$.

We can apply this theorem and show that the (φ, Γ) -module associated with $\rho_{?|G_{\mathbb{Q}_p}}$ is trianguline. We explicit now the triangulation, given in [HJ13, §3.3].

As seen before, a p -stabilization of \mathfrak{p} corresponds to a permutation ν and a map ε .

The eigenvalues of φ are given by

$$\begin{aligned} & \prod_{\tau \in I_{\mathfrak{p}}} \tau(\varpi_{\mathfrak{p}})^{c_1 + \mu_{\tau,1}} \beta_{\mathfrak{p},1}, \\ & \prod_{\tau \in I_{\mathfrak{p}}} \tau(\varpi_{\mathfrak{p}})^{c_i + \mu_{\tau,i}} \frac{\beta_{\mathfrak{p},i-1}}{\beta_{\mathfrak{p},i}}, \\ & \prod_{\tau \in I_{\mathfrak{p}}} \tau(\varpi_{\mathfrak{p}})^{c_g + \mu_{\tau,g}} \frac{\beta_{\mathfrak{p},g-1}}{\beta_{\mathfrak{p},g}^2} \end{aligned}$$

where c_i 's are a positive integer independent of the weight.

We define the following characters of $F_{\mathfrak{p}}$ with value in $\mathcal{O}(\mathcal{X})$:

$$\begin{aligned} \chi_{\mathfrak{p},1}(\varpi_{\mathfrak{p}}) &= \beta_{\mathfrak{p},1}, \\ \chi_{\mathfrak{p},i}(\varpi_{\mathfrak{p}}) &= \frac{\beta_{\mathfrak{p},i-1}}{\beta_{\mathfrak{p},i}}, \\ \chi_{\mathfrak{p},g}(\varpi_{\mathfrak{p}}) &= \frac{\beta_{\mathfrak{p},g-1}}{\beta_{\mathfrak{p},g}^2}, \end{aligned}$$

and $\chi_{\mathfrak{p},1}(u) = \prod_{\tau \in I_{\mathfrak{p}}} \tau(u)^{c_i + \mu_{\tau,i}}$.

From [HJ13, Lemma 19] we have that the graded pieces of $\mathbf{D}_{\text{rig}}^\dagger(V_{\text{sta}|_{\mathfrak{p}}})$ are then given by the characters $\chi_{\mathfrak{p},g}, \dots, 1, \dots, \chi_{\mathfrak{p},g}^{-1}$.

Concerning V_{spin} , we number the subsets of $\{1, \dots, g\}$ as I_1, I_2, \dots, I_{2g} . Each I_j correspond to a map $\varepsilon_j : \{1, \dots, g\} \rightarrow \pm 1$.

We have then the graded pieces $\delta_{\mathfrak{p},j}$ are given by the characters

$$\begin{aligned} \delta_{\mathfrak{p},\varepsilon_j}(u) &= \prod_{\tau \in I_{\mathfrak{p}}} \tau(u)^{d_j + \frac{k_0 + \sum_i \varepsilon_j(i) k_{\tau,i}}{2}}, \\ \delta_{\mathfrak{p},\varepsilon_j}(\varpi_{\mathfrak{p}}) &= \beta_{\mathfrak{p},g} \prod_{i \in I_j} \chi_{\mathfrak{p},i}(\varpi_{\mathfrak{p}}). \end{aligned}$$

Let V be either V_{sta} or V_{spin} . If $\pi_{\mathfrak{p}}$ is Steinberg, there is only one choice of a regular (φ, N) -sub-module $D_{\mathfrak{p}}$ of $\mathbf{D}_{\text{st}}(V_{G_{F_p}})$, where V is one of the two representations associated with π described above. If the form is not Steinberg at \mathfrak{p} many different regular sub-module can be chosen.

In any case, we expect (and we shall assume in the follow) that there is at most one trivial zero for each \mathfrak{p} . Consider now the representation π of parallel weight \underline{k} (i.e. associated with $N_{F/\mathbb{Q}}(\det^{\underline{k}})$, $\underline{k} \in \mathbb{Z}$) as in the introduction.

We give a preliminary proposition on the factorization of the \mathcal{L} -invariant. Recall the set $S^{\text{Sph},1}$ and S^{Stb} defined in the introduction, we have the following;

Proposition 4.14. *We have the following factorization*

$$\mathcal{L}(V, D) = \mathcal{L}(V, D)^{\text{Sph}} \prod_{\mathfrak{p} \in S^{\text{Stb}}} \mathcal{L}(V, D)_{\mathfrak{p}},$$

where $\mathcal{L}(V, D)^{\text{Sph}}$ comes from the prime in S^{Sph} and the factors $\mathcal{L}(V, D)_{\mathfrak{p}}$ are local.

Proof. We follow [Hid07, §1.3]. In the notation of Section 3, we write $W_1 = \oplus_{\mathfrak{p} \in S^{\text{Stb}}} W_{\mathfrak{p},1}$ and $M_1 = \oplus_{\mathfrak{p} \in S^{\text{Sph},1}} M_{\mathfrak{p},1}$. We are left to show that the endomorphism $\iota_f \circ \iota_c^{-1}$ of $\mathcal{D}_{\text{cris}}(W_1 \oplus M_1) \cong E^t$ keeps stable $\mathcal{D}_{\text{cris}}(M_1)$ and on the quotient it respects the direct sum decomposition $\oplus_{\mathfrak{p} \in S^{\text{Stb}}} \mathcal{D}_{\text{cris}}(W_{\mathfrak{p},1})$.

Consider a prime $\mathfrak{p}_0 \in S^{\text{Stb}}$ and a cocycle $c \in H^1(V, D)$ such that $\text{res}_{\mathfrak{p}}(c) = 0$ for all $\mathfrak{p} \neq \mathfrak{p}_0$. This means that $\text{res}_{\mathfrak{p}}(c) = 0 \in H_f^1(F_{\mathfrak{p}}, V) = H_f^1(F_{\mathfrak{p}}, M_{\mathfrak{p}})$ (by (3.3)). Hence by Proposition 2.10 (which holds only for \mathfrak{p} in S^{Stb}) we have $\iota_{c,\mathfrak{p}}(c) = \iota_{f,\mathfrak{p}}(c) = 0$.

We have also $\iota_{c,\mathfrak{p}}(c) = 0$ for all primes $\mathfrak{p} \neq \mathfrak{p}_0$ as H_c^1 is the direct sum complement of H_f^1 (see [Ben11, Proposition 1.5.9]).

The proposition then follows from standard linear algebra as in [Hid07, Corollary 1.9]. \square

Remark 4.15. *A key ingredient in the proof of the factorization at Steinberg places is that each prime ideal brings a single trivial zero.*

We consider now the case $V = V_{\text{sta}}$. We have a contribution to trivial zeros from the $\pi_{\mathfrak{p}}$'s which are Steinberg and possibly from the $\pi_{\mathfrak{p}}$ which are spherical. In particular, if we choose the regular sub-module coming from an ordinary filtration, we always have a trivial zero coming from each place.

For all $1 \leq s \leq \min(k - g - 1, g - 1)$ we have also e_{Stb} trivial zeros for $V(s)$.

Theorem 4.16. *For $\pi_{\mathfrak{p}}$ Steinberg we have*

$$\mathcal{L}(V, D)_{\mathfrak{p}} = -\frac{1}{f_{\mathfrak{p}}} \frac{\text{d log}_p \beta_{\mathfrak{p},1}(k)}{\text{d}k} \Big|_{k=\underline{k}},$$

where k is the parallel weight variable.

For $1 \leq s \leq \min(k - g - 1, g - 2)$ we also have

$$\mathcal{L}(V(s), D(s))_{\mathfrak{p}} = -\frac{1}{f_{\mathfrak{p}}} \frac{\text{d log}_p(\beta_{\mathfrak{p},s-1} \beta_{\mathfrak{p},s}^{-1}(k))}{\text{d}k} \Big|_{k=\underline{k}}$$

and if $g - 1 \leq k - g - 1$ we have

$$\mathcal{L}(V(g-1), D(g-1))_{\mathfrak{p}} = -\frac{1}{f_{\mathfrak{p}}} \frac{\text{d log}_p(\beta_{\mathfrak{p},g-1} \beta_{\mathfrak{p},g}^{-2}(k))}{\text{d}k} \Big|_{k=\underline{k}}.$$

Proof. We apply Theorem 4.1 for the $f_{\mathfrak{p}}$ -th root of $\chi_{\mathfrak{p},i}(\varpi_{\mathfrak{p}})$ and we note that we can specialize to a parallel family, so that no contribution from the denominator appears. The $\log_p(u)$ disappears because of the change of variable $T \mapsto u^k - 1$ (u any topological generator of \mathbb{Z}_p^{\times}). \square

Remark 4.17. *The presence of $f_{\mathfrak{p}}$ in the denominator is explained in term of L -function and Euler factors at p in [Hid09].*

From now on, $V = V_{\text{spin}}(k-1)$ ($s = k-1$ is the only critical integer); if $\pi_{\mathfrak{p}}$ is spherical it should not give any trivial zeros (as the corresponding p -adic representation is conjectured to be crystalline and consequently the β_i 's are Weil numbers of non-zero weight).

So we are left to see what happen at the primes Steinberg at \mathfrak{p} . Twisting by $\beta_{\mathfrak{p},g}$ the triangulated (φ, Γ) -module of ρ_{spin} we are in the hypothesis of Theorem 4.1 and we have

Theorem 4.18. *For $\pi_{\mathfrak{p}}$ Steinberg we have*

$$\mathcal{L}(V, D)_{\mathfrak{p}} = -\frac{1}{f_{\mathfrak{p}}} \frac{d \log_p \beta_{\mathfrak{p},1}(k)}{dk} \Big|_{k=\underline{k}},$$

where k is a parallel weight variable.

5 The case of the adjoint representation

We prove Theorem 1.4 of the introduction. We consider only the case $g = 2$. Fix an automorphic representation π of weight $\underline{k} = (\underline{k}_{\tau,1}, \dots, \underline{k}_{\tau,g}; \underline{k}_0)_{\tau}$ and let $V = V_{\text{spin}}$ be the spin representation associated with π . Let $\rho = \rho_{\text{spin}}$ be the corresponding big Galois representation.

We specialize the eigenvariety \mathcal{X} of Theorem 4.11 to the subspace of the weight space given by the equations $k_{\tau,i} = k_{\tau',i}$ if τ and τ' induce the same p -adic place \mathfrak{p} and $k_0 = \underline{k}_0$. We shall denote the new variable by $k_{\mathfrak{p},i}$ and this eigenvariety by \mathcal{X}' . For simplicity, we rewrite the graded pieces of V as

$$\begin{aligned} \delta_{\mathfrak{p},1}(\varpi_{\mathfrak{p}}) &= F_{\mathfrak{p},1}^{-1}(k), \quad \delta_{\mathfrak{p},1}(u) = N_{F_{\mathfrak{p}}/\mathbb{Q}_p}(u)^{\frac{k_0+k_{\mathfrak{p},1}+k_{\mathfrak{p},2}-3}{2}}, \\ \delta_{\mathfrak{p},2}(\varpi_{\mathfrak{p}}) &= F_{\mathfrak{p},2}^{-1}(k), \quad \delta_{\mathfrak{p},2}(u) = N_{F_{\mathfrak{p}}/\mathbb{Q}_p}(u)^{\frac{k_0+k_{\mathfrak{p},2}-k_{\mathfrak{p},1}+1}{2}}, \\ \delta_{\mathfrak{p},3}(\varpi_{\mathfrak{p}}) &= F_{\mathfrak{p},3}(k), \quad \delta_{\mathfrak{p},3}(u) = N_{F_{\mathfrak{p}}/\mathbb{Q}_p}(u)^{\frac{k_0-k_{\mathfrak{p},2}+k_{\mathfrak{p},1}-1}{2}}, \\ \delta_{\mathfrak{p},4}(\varpi_{\mathfrak{p}}) &= F_{\mathfrak{p},4}(k), \quad \delta_{\mathfrak{p},4}(u) = N_{F_{\mathfrak{p}}/\mathbb{Q}_p}(u)^{\frac{k_0-k_{\mathfrak{p},1}-k_{\mathfrak{p},2}+3}{2}} \end{aligned}$$

where $k = (k_{\mathfrak{p},1}, k_{\mathfrak{p},2}; k_0)_{\mathfrak{p}}$.

The representation space of $\text{Ad}(V)$ is given by the matrices

$$\mathfrak{S}_{\mathfrak{p}_4} = \{X \in \mathfrak{S}_{\mathfrak{L}_4} | XJ^t + JX = 0\}.$$

The p -stabilization on V induces a natural p -stabilization and consequently a regular sub-module D_{Ad} on $\text{Ad}(V_{\text{spin}})$. We have

$$\begin{aligned} D_{\text{Ad}-1} &= \{\text{nilpotent } X\}, \\ D_{\text{Ad}0} &= \{\text{unipotent } X\}. \end{aligned}$$

The basis for the space $D_{\text{Ad}0}/D_{\text{Ad}-1}$ is given by the two diagonal matrices $d_1 = [-1, 0, 0, 1]$ and $d_2 = [0, -1, 1, 0]$. We shall denote by $d_{\mathfrak{p},i}$ these matrices when seen as a vector for $\text{Ad}(V_{\mathfrak{p}})$.

Proposition 5.1. *Suppose that **C1-C4** holds for V . Suppose that the classical E -point x in the eigenvariety \mathcal{X}' corresponding to π is étale above the weight space. Then, the space $\mathcal{L}(D_{\text{Ad}}, V)$ is generated by the image of $\left(\frac{d \log_p \delta_{\mathfrak{p},i}}{dk_{\mathfrak{p}',j}} d_{\mathfrak{p},i}\right)_{\mathfrak{p}',j=1,2}$.*

Proof. The proof is standard and goes back to [MT90], so we shall only sketch it. Let $A = E[T]/(T^2)$. Consider an infinitesimal deformation of ρ given by

$$\rho_A = V \oplus \rho';$$

note that ρ' can be written as the first order truncation of $\frac{\partial \rho}{\partial v}$, where v is any direction in the weight space. From ρ_A we can construct a cocycle $c_{x,A}$ defined by

$$G_F \ni \sigma \mapsto \rho'(\sigma)V^{-1}(\sigma).$$

It is easy to check that this defines a cocycle with values in $V \otimes V^*$. Moreover its image lands in $\text{Ad}(V) \subset V \otimes V^*$ as the determinant is fixed (by our choice of the Hodge-Tate weight on \mathcal{X}'). Writing explicitly the matrix for the (φ, Γ) -module associated with ρ_A we obtain

$$\left(\begin{array}{cccc} \frac{\partial \delta_{p,1}}{\partial v} & * & * & * \\ & \frac{\partial \delta_{p,2}}{\partial v} & * & * \\ & & \frac{\partial \delta_{p,3}}{\partial v} & * \\ & & & \frac{\partial \delta_{p,4}}{\partial v} \end{array} \right)_{|k=\underline{k}} \left(\begin{array}{cccc} \delta_{p,1}^{-1} & * & * & * \\ & \delta_{p,2}^{-2} & * & * \\ & & \delta_{p,3}^{-1} & * \\ & & & \delta_{p,4}^{-1} \end{array} \right)_{|k=\underline{k}}$$

In particular, they are upper triangular and their projection via ι_f onto the vector $d_{p,1}$ is $\frac{d \log_p F_{p,1}(k)}{dv} \Big|_{k=\underline{k}}$. Similarly for $d_{p,2}$.

We also have that the projection via ι_c onto $d_{p,1}$ is $-\frac{\partial(k_{p,1}+k_{p,2})/2}{\partial v} \Big|_{k=\underline{k}}$.

This cocycle lies $H^1(G_{F,S}, V \otimes V^*)$ by construction of ρ .

As $\left\{ \frac{\partial}{\partial k_{p,i}} \right\}_{p,i=1,2}$ is a base of the tangent space at x in \mathcal{X}' we are done. \square

We can now prove Theorem 1.4 which we recall now;

Theorem 5.2. *We have*

$$\mathcal{L}(\text{Ad}(V_{\text{spin}}), D_{\text{Ad}}) = \prod_{\mathfrak{p}} \frac{2}{f_{\mathfrak{p}}^2} \det \left(\begin{array}{cc} \frac{\partial \log_p F_{p,i,1}(k)}{\partial k_{p,j,1}} & \frac{\partial \log_p F_{p,i,2}(k)}{\partial k_{p,j,1}} \\ \frac{\partial \log_p F_{p,i,1}(k)}{\partial k_{p,j,2}} & \frac{\partial \log_p F_{p,i,2}(k)}{\partial k_{p,j,2}} \end{array} \right)_{1 \leq i,j \leq t \mid k=\underline{k}}.$$

Proof. Once we have Proposition 5.1, we just have to follow the proof of [Hid06, Theorem 3.73]. The matrix of ι_f is exactly what appears in the Theorem, while the matrix of ι_c can be directly calculated using the formula $\frac{d \log(u^{\pm k_{p,i}})}{dk_{p',j}} = \pm \delta_{p,p'} \delta_{i,j}$ (where $\delta_{a,b}$ here is Kronecker delta) and gives a contribution of 2^{-1} for each prime ideal \mathfrak{p} . \square

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